# Phase Lead S-Plane Compensation

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**A new technique for designing a series phase lead compensation network is presented in this paper. It is an extension of a known S-plane design technique; however, contrary to the known technique which is a trial and error procedure, the new technique is straightforward. Given the open-loop transfer function of the uncompensated system, the desired location of the two dominant complex poles of the closed-loop system and the desired open-loop zero frequency gain, two quantities can be found either analytically or graphically. It is shown that, using these two quantities, the values of the pole and the zero of the compensation network can be found either analytically or graphically. A geometric construction to accomplish the latter is presented and proven. The new technique is demonstrated by two examples which are solved using Bode plots. A mathematical algorithm suitable for a computer program which implements the new technique is also given in this paper.**

## Nomenclature

| | |
|---|---|
| $a$ | = distance between the zero of the compensator and a dominant closed-loop complex pole |
| $b$ | = distance between the pole of the compensator and a dominant closed-loop complex pole |
| $C$ | = ratio between a product of zeroes and a product of poles |
| $D$ | = ratio between products of length of vectors in the $S$-plane |
| $G, G'$ | = transfer functions |
| $h, h_{P_i}, h_{Z_j}$ | = imaginary parts of points in the $S$-plane |
| $i, j, k$ | = running indices |
| $K, K_0$ | = gains |
| $m, n$ | = limits of running indices |
| $P$ | = point in the $S$-plane |
| $P_i$ | = $i$th pole of an open-loop transfer function |
| $Q$ | = $DC/K_0$ |
| $R$ | = point in the $S$-plane |
| $S$ | = Laplace transform variable |
| $S_1$ | = dominant closed-loop complex pole |
| $S_0$ | = zero of the compensator |
| $S_x$ | = pole of the compensator |
| $x, y$ | = lengths in the $S$-plane |
| $T, U, V, W, Z$ | = points in the $S$-plane |
| $Z_j$ | = $j$th zero of an open-look transfer function |
| $\phi, \psi_0, \psi_{10}, \psi_{1000}, \psi_x$ | = angles in the $S$-plane |
| $\alpha, \delta$ | = angles in the $S$-plane |
| $\zeta$ | = damping factor |
| $\mu$ | = order of the pole (or zero) of the open-loop transfer function at the origin |
| $\lambda$ | = angle in the $S$-plane |
| $\rho_{P1}, \rho_{P2}, \rho_{P3}, \rho_0, \rho_{10}, \rho_{1000}$ | = length of vectors in the $S$-plane |
| $\sigma, \sigma_{P_i}, \sigma_{Z_j}$ | = real parts of points in the $S$-plane |
| $\varphi_{P1}, \varphi_{P2}, \varphi_{P3}, \chi$ | = angles in $S$-plane |
| $\omega_n$ | = natural frequency of the closed-loop system |

## Introduction

ONE of the most frequent design problems a control engineer is faced with is the design of a compensation network. Usually it is a series phase lead network which is attempted first and only in cases where such a network is not suitable the designer tries other means. When a phase lead network is used, the rise time of the compensated system is smaller than that of the uncompensated system and at the same time the overshoot is smaller too. For this reason a phase lead compensation is usually preferable,[5] particularly in aerospace system.[1,2] The most common technique for designing a phase lead compensator is the frequency response technique on the Bode plot.[3-5] Another attractive technique is that of compensation on the $S$-plane[6-8] which utilizes Root Locus methods. As presented in Refs. 6–8 this technique forces the user to resort to trial and error cycles. Such cycles are needed because the open-loop gain of the compensated system is not a specification used explicitely in the design cycle but rather is a result obtained at the end of each cycle. Therefore, if the resultant gain does not meet the design specifications a new guess concerning the compensation network has to be made and a new cycle has to be performed. This goes on until an appropriate gain is achieved.

In this paper a design technique is introduced in which trial and error is eliminated. As might be expected the use of this technique is meaningful only in the cases in which the use of the old technique[6-8] is meaningful; i.e., in cases where the closed-loop system possesses two dominant poles.

## Technique

For the class of systems under consideration it is possible to express the required transient behavior of the system by means of a pair of dominant closed-loop complex poles. In addition, the desired steady-state behavior of the system can be expressed by $K_0$, the open-loop system zero frequency gain.

Suppose that the open loop system transfer function is given by

$$G'(S) = K_0 \cdot S^{\mu} \prod_{j=1}^{m} \left(\frac{S}{Z_j} + 1\right) \Big/ \prod_{i=1}^{n} \left(\frac{S}{P_i} + 1\right) \tag{1}$$

where $S^{\mu}$ is the $\mu$th-order zero or pole at the origin, $Z_j$, $j = 1, 2, 3, \ldots, m$ are the zeroes of the system and $P_i$, $i = 1, 2, 3, \ldots, n$ are the poles of the system. The open-loop transfer function of the series phase lead compensated network may be written as

$$G(S) = G'(S) \frac{S/S_0 + 1}{S/S_x + 1} \tag{2}$$

where $S_0$ is the zero of the compensation network,† $S_x$ is its pole and $S_0 < S_x$.

Let $S_1$ be one of the two dominant complex poles of the

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† From now on the compensation network will be assumed to be a phase lead network unless otherwise specified.

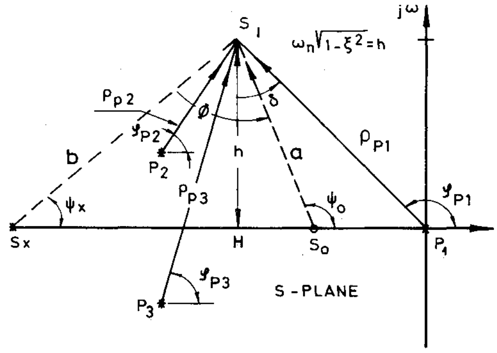


**Fig. 1** *S*-**plane representation of a phase lead compensation problem.**

closed-loop compensated system then, from the Root Locus rules, the following exist

$$K_0|S_1|^{\mu}\frac{\prod_{i=1}^{n}P_i}{\prod_{j=1}^{m}Z_j}\cdot\frac{\prod_{j=1}^{m}|S_1+Z_j|}{\prod_{i=1}^{n}|S_1+P_i|}\cdot\frac{S_x}{S_0}\cdot\frac{|S_1+S_0|}{|S_1+S_x|}=1 \qquad (3)$$

and

$$\mu\,\angle S_1+\prod_{j=1}^{m}\angle S_1+Z_j+\angle S_1+S_0-\left(\prod_{i=1}^{n}\angle S_1+P_i+\angle S_1+S_x\right)=$$
$$(2K+1)\pi \qquad k=0,\pm1,\pm2,\ldots \qquad (4)$$

Denote

$$a=|S_1+S_0| \qquad (5a)$$
$$b=|S_1+S_x| \qquad (5b)$$
$$\psi_0=\angle S_1+S_0 \qquad (5c)$$
$$\psi_x=\angle S_1+S_x \qquad (5d)$$
$$D=|S_1|^{-\mu}\frac{\prod_{i=1}^{n}|S_1+P_i|}{\prod_{j=1}^{m}|S_1+Z_j|} \qquad (5e)$$
$$C=\prod_{j=1}^{m}Z_j\Big/\prod_{i=1}^{n}P_i \qquad (5f)$$

then Eqs. (3) and (4) can be written as follows

$$a/b=D\cdot C\cdot(1/K_0)\cdot S_0/S_x \qquad (6)$$

and

$$\psi_0-\psi_x=(2k+1)\pi-\mu\angle S_1-\sum_{j=1}^{m}\angle S_1+Z_j+\sum_{i=1}^{n}\angle S_1+P_i \qquad (7)$$
$$k=0,\pm1,\pm2,\ldots$$

We refer to Fig. 1 in order to illustrate the geometrical problem involved in the $S$-plane design. The open-loop transfer function of the system which is chosen for this illustration has a pole $P_1$ at the origin, two complex conjugate poles one at $P_2$ and one at $P_3$ and no zeroes in the finite part of the $S$-plane (i.e., $\mu=0$). The quantity $D$ defined by Eq. (5e) is a measurable quantity and is given by

$$D=\rho_{P_1}(\rho_{P_2}\cdot\rho_{P_3})/1 \qquad (8)$$

From Eq. (5f) $C$ can be computed by

$$C=1/(P_2\cdot P_3) \qquad (9)$$

while $K_0$ is a given quantity. In the triangle $S_0S_1S_x$ the angle is given by

$$\phi=\psi_0-\psi_x \qquad (10)$$

[It should be noted that Eq. (10) does not depend on the particular transfer function which is chosen for this illustration.] Equation (7) can be written in this case as

$$\phi=-180°+(\varphi_{P1},\varphi_{P2},\varphi_{P3})° \qquad (11)$$

Using Eqs. (8) and (9), Eq. (6) can be written as

$$a/b=[(\rho_{P1}\cdot\rho_{P2}\cdot\rho_{P3})/(P_1\cdot P_2\cdot K_0)]\cdot S_0/S_x \qquad (12)$$

The problem of designing a compensator has been transformed into the problem of constructing the triangle $S_0S_1S_x$ whose apex is at $S_1$ and which in this example satisfies Eqs. (11) and (12). In general the problem is to construct the triangle $S_0S_1S_x$ whose height and apex angle are known and which satisfies Eq. (6). Denoting

$$Q=D\cdot C\cdot 1/K_0 \qquad (13)$$

Equation (6) can be written as

$$a/b=Q\,S_0/S_x \qquad (14)$$

Referring to Fig. 2 it can be seen that

$$h=a\cdot\cos\alpha \qquad (15)$$
$$h=b\cdot\cos(\phi-\alpha) \qquad (16)$$

therefore

$$\frac{a}{b}=\frac{\cos\phi\cdot\cos\alpha+\sin\phi\cdot\sin\alpha}{\cos\alpha} \qquad (17)$$

or

$$a/b=\cos\phi+\sin\phi\cdot\tan\alpha \qquad (18)$$

It can also be seen in Fig. 2 that

$$\frac{S_0}{S_x}=\frac{\sigma-h\cdot\tan\alpha}{\sigma+h(\sin\phi-\cos\phi\cdot\tan\alpha)/(\cos\phi+\sin\phi\cdot\tan\alpha)} \qquad (19)$$

Substituting Eqs. (18) and (19) into Eq. (14) yields

$$\cos\phi+\sin\phi\cdot\tan\alpha=$$
$$Q\cdot\frac{\sigma-h\cdot\tan\alpha}{\sigma+h(\sin\phi-\cos\phi\cdot\tan\alpha)/(\cos\phi+\sin\phi\cdot\tan\alpha)} \qquad (20)$$

hence

$$\sigma\cdot(\cos\phi+\sin\phi\cdot\tan\alpha)+h(\sin\phi-\cos\phi\cdot\tan\alpha)=$$
$$Q\sigma-Qh\cdot\tan\alpha \qquad (21)$$

and after rearranging terms and factoring $\tan\alpha$ out, Eq. (21) yields

$$\tan\alpha=\frac{\sigma(Q-\cos\phi)-h\cdot\sin\phi}{\sigma\sin\phi+h(Q-\cos\phi)} \qquad (22)$$

When the numerator and the denominator of Eq. (22) are divided by $h(Q-\cos\phi)$ the following equation is obtained:

$$\tan\alpha=\frac{\sigma/h-\sin\phi/(Q-\cos\phi)}{1+(\sigma/h)\cdot\sin\phi/(Q-\cos\phi)} \qquad (23)$$

From Fig. 2 it can be seen that

$$\sigma/h=\tan\delta \qquad (24)$$

then defining an angle $\lambda$ by

$$\tan\lambda\triangleq\sin\phi/(Q-\cos\phi) \qquad (25)$$

Equation (23) can be written as

$$\tan\alpha=\frac{\tan\delta-\tan\lambda}{1+\tan\delta\cdot\tan\lambda} \qquad (26)$$

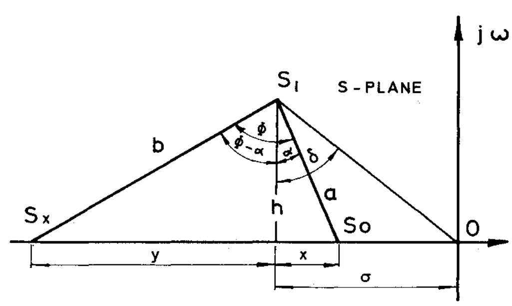


**Fig. 2 Geometric representation of the compensation network.**

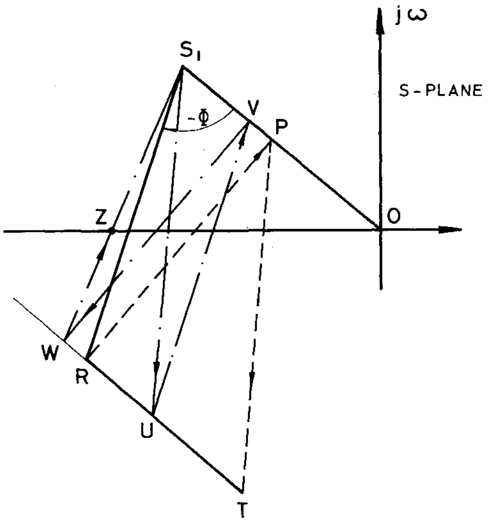


**Fig. 3 A construction method to find the location of the zero of the compensation network.**

or

$$\tan\alpha = \tan(\delta - \lambda)$$

therefore

$$\alpha = \delta - \tan^{-1}[\sin\phi/(Q-\cos\phi)] \tag{27}$$

A simple geometric method to find $S_0$ and $S_x$ will be now presented. It is based on Eq. (27) and will be proven later. Referring to Fig. 3 and noting that positive angles are drawn anticlockwise, one has to carry out the following steps. a) Locate the desired pole $S_1$ in the $S$-plane and draw the line $OS_1$. b) Draw the line $S_1R$ at an angle $-\phi$ from $S_1O$ and at a length of $n$ units ($n$ is arbitrarily chosen to give the construction reasonable dimensions). c) Draw a line $RT$ parallel to $S_1O$ at a length of $nQ$ units. d) Draw the line $RP$ perpendicular to $S_1O$. e) Draw the line $PT$. f) Draw $S_1U$ parallel to $PT$. g) Draw $UV$ parallel to $RS_1$. h) Draw $VW$ parallel to $PR$. i) Draw a line between $W$ and $S_1$. Its intersection $Z$ with the real axis is the point $S_0$.

The validity of this geometrical method will be now proven. Since $S_1R = n$ (construction) and since $RP \parallel S_1O$ then

$$S_1P = n\cdot\cos\phi \tag{28}$$

and

$$RP = n\cdot\sin\phi \tag{29}$$

Since $S_1U \parallel PT$ and $RT \parallel S_1O$ then

$$UT = S_1P = n\cdot\cos\phi \tag{30}$$

and because $RT = nQ$ (construction) then

$$RU = RT - UT = nQ - n\cdot\cos\phi \tag{31}$$

As $UV \parallel RS_1$ and $RT \parallel S_1O$ then

$$S_1V = RU = n(Q-\cos\phi) \tag{32}$$

Because $WV \parallel S_1O$ then

$$\tan(\measuredangle VS_1W) = WV/S_1V \tag{33}$$

but since $WV \parallel RP$ and $RT \parallel S_1O$ then

$$WV = RP \tag{34}$$

Using Eq. (29) in Eq. (34) yields

$$WV = n\cdot\sin\phi \tag{35}$$

Substituting Eqs. (32) and (35) in Eq. (33) yields

$$\tan(\measuredangle VS_1W) = \sin\phi/(Q-\cos\phi) \tag{36}$$

therefore in view of the definition of Eq. (25)

$$\measuredangle VS_1W = \lambda$$

Finally referring to Fig. 4, in which only a part of the details of Fig. 3 are redrawn, one realizes that

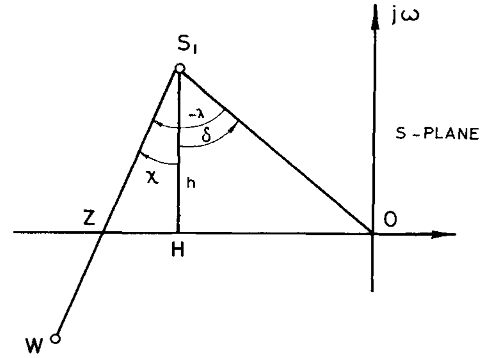


**Fig. 4 The relationship between the angles $\chi$, $\delta$ and $\lambda$.**

$$\chi \equiv \alpha \tag{37}$$

thus from Fig. 2

$$Z = S_0 \tag{38}$$

The design of the compensator can, of course, be carried out in an entirely analytical way by means of a computer program. For this purpose Eqs. (5e) and (5f) can be written as

$$D = (\sigma^2 + h^2)^{-\mu/2} \frac{\prod_{i=1}^{n} [(\sigma+\sigma_{P_i})^2 + (h+h_{P_i})^2]^{1/2}}{\prod_{j=1}^{m} [(\sigma+\sigma_{Z_j})^2 + (h+h_{Z_j})^2]^{1/2}} \tag{39}$$

and

$$C = \frac{\prod_{j=1}^{m} (\sigma_{Z_j}{}^2 + h_{Z_j}{}^2)^{1/2}}{\prod_{i=1}^{n} (\sigma_{P_i}{}^2 + h_{P_i}{}^2)^{1/2}} \tag{40}$$

where $\sigma_{Z_j}$ is the real part of the $j$th zero, $h_{Z_j}$ is the imaginary part of the $j$th zero, $\sigma_{P_i}$ is the real part of the $i$th pole, and $h_{P_i}$ is the imaginary part of the $i$th zero.

$Q$ is found using Eq. (13) and for $\phi$ Eq. (7) is used in the following form:

$$\phi = (2k+1)\pi - \mu\cdot\tan^{-1}\frac{h}{\sigma} - \sum_{j=1}^{m}\tan^{-1}\frac{h+h_{Z_i}}{\sigma+\sigma_{Z_j}} + \sum_{i=1}^{n}\tan^{-1}\frac{h+h_{P_i}}{\sigma+\sigma_{P_i}} \tag{41}$$

To find $\alpha$ one can use the equation

$$\alpha = \tan^{-1}(\sigma/h) - \tan^{-1}[\sin\phi/(Q-\cos\phi)] \tag{42}$$

which is essentially Eq. (27). Finally to find $S_0$ and $S_x$ one can use the following equations

$$S_0 = \sigma - h\cdot\tan\alpha \tag{43}$$

$$S_x = \sigma + h\cdot\tan(\phi-\alpha) \tag{44}$$

## Example 1

To demonstrate the method an example is chosen which has been already solved using the frequency response (Bode) method. This is example 10-1 on page 439 of Ref. 5. The system presented in that example has the open-loop transfer function

$$G(S) = K_0/S(S+1) \tag{45}$$

The specifications of the closed-loop system are 1) The phase margin of the system must be at least 45°. 2) When the input to the system is a ramp function the steady-state error of the output variable should be less than 0.1° per deg/sec of the final output velocity. The second requirement implies that

$$K_0 = 10\ \text{sec}^{-1}$$

It can be shown that the system, whose open-loop transfer function is given by Eq. (45), whose gain $K_0$ is equal to 10 and

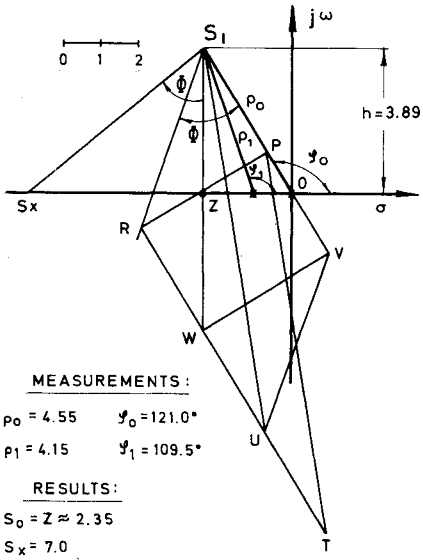


**Fig. 5 A graphical solution of example 1.**

whose phase margin due to phase lead compensation is 45°, has a pair of complex poles at $-2.36 \pm j3.89$. That is, $\sigma = 2.36$ and $h = 3.89$.

The construction from which $S_0$ and $S_x$ are determined is shown in Fig. 5. First the lines $\rho_0$ and $\rho_1$ are drawn, respectively, from the open loop poles at 0 and at $-1$ to $S_1$ located at $-2.36+j3.89$. It is determined by measurements that $\rho_0 = 4.55$, $\rho_1 = 4.15$, $\varphi_0 = 121.0°$ and $\varphi_1 = 109.5°$. Therefore

$$D = \rho_0 \cdot \rho_1 / K_0 = 4.55 \cdot 4.15/10 = 1.89$$

and

$$\phi = 180° - (121.0 + 109.5)° = 50.5°$$

It is obvious that in this case

$$Q = D = 1.89$$

$S_1R$ is constructed in such a direction as to make $\measuredangle 0S_1R = 50.5°$. The value of $n$ is chosen to be 5 and therefore the length of $S_1R$ is 5 units. $RT$ is constructed parallel to $S_1O$ at a length of $n \cdot Q = 5 \cdot 1.89$ units. $RP$ is drawn perpendicular to $OS_1$, $S_1U$ parallel to $PT$, $UV$ parallel to $RS_1$, $VW$ parallel to $PR$ and finally $Z$, which is $S_0$, is obtained by the intersection of $WS_1$ with the real axis. $S_x$ is found by drawing $\measuredangle ZS_1S_x$ at an angle of $-50.5°$. The results are

$$S_0 = 2.35$$
$$S_x = 7.0$$

which check with

$$S_0 = 2.4$$
$$S_x = 7.2$$

obtained in the example of Ref. 5 by the use of Bode plots.

## Example 2

This example is also taken from Ref. 5. It is example 10-2 on page 444. The open-loop transfer function of the system is

$$G(S) = \frac{K}{S(1+0.1S)(1+0.001S)} \tag{46}$$

The specifications of the closed-loop system are 1) $K = 1000$. 2) The natural frequency corresponding to the two dominant complex poles is $\omega_n = 209.3$. 3) The damping factor of these poles is $\zeta = 0.637$. These requirements are the $S$-plane equivalent of the zero frequency gain and the phase margin specifications stated in Ref. 5.

From $\omega_n$ and $\zeta$ it is found that $\sigma = 133.34$ and that $h = 161.4$. Plotting the vectors from the poles of Eq. (46) to $S_1$ (Fig. 6) and measuring their lengths and phases one obtains

$$\rho_0 = 210, \ \rho_{10} = 205, \ \rho_{1000} = 880$$

and

$$\psi_0 = 129.5°, \ \psi_{10} = 127.4°, \ \psi_{1000} = 10.5°$$

Thus

$$\phi = -180° + (\psi_0 + \psi_{10} + \psi_{1000})° = 86.9° \tag{47a}$$
$$D = \rho_0 \cdot \rho_{10} \cdot \rho_{1000} = 37884 \cdot 10^3 \tag{47b}$$
$$C = 1/10 \cdot 1000 = 10^{-4} \tag{47c}$$

and

$$Q = (1/K_0)DC = 3.7884 \tag{47d}$$

Choosing $n = 200$ and following the graphical steps of the technique, one obtains the results

$$S_0 \approx 63$$
$$S_x \approx 437$$

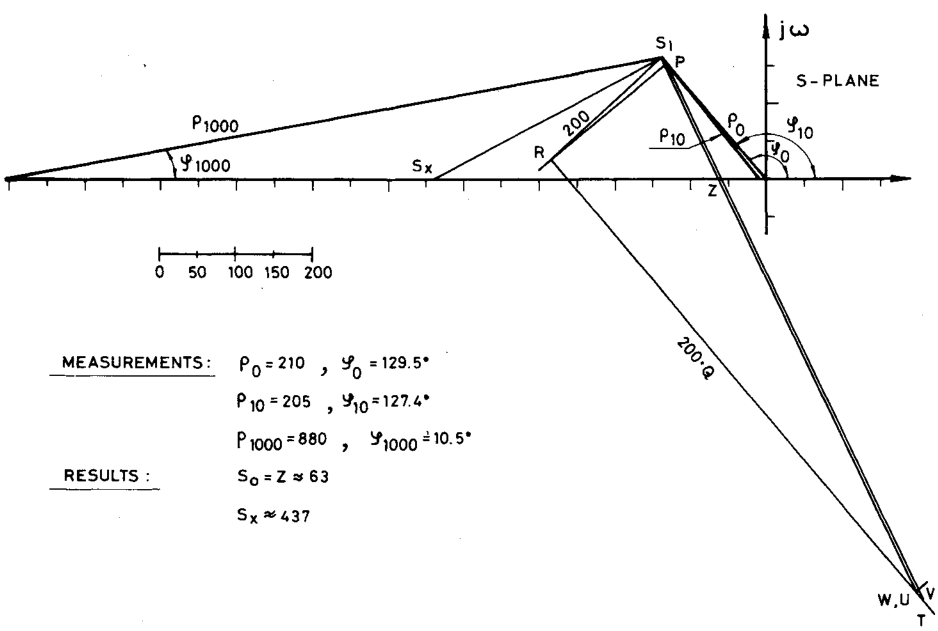


**Fig. 6 A graphical solution of example 2.**

Using a desk calculator, though, to compute Eq. (42) one obtains

$$\alpha = 24.45^\circ$$

which when used to compute $S_0$ and $S_x$ [using Eqs. (43) and (44)] yields

$$S_0 = 59.95$$
$$S_x = 450.35$$

The results of Ref. 5 are

$$S_0 = 59.88$$
$$S_x = 450.45$$

It is concluded that the graphical method is accurate enough.

## Conclusions

A new technique for designing a phase lead compensator on the $S$-plane has been presented in this work. This technique eliminates the need for trial and error cycles which are a feature of the present technique.[6-8] It can be carried out in part through a graphical construction or in an entirely analytical way. A digital computer program which implements the new technique can be written using the algorithm given by Eqs. (39–44).

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# Vibration and Buckling of Shells under Initial Stresses

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**The force-displacement matrix equations are formulated for a 48 degrees-of-freedom doubly curved shell finite element. The formulations include, in addition to stiffness, the effect of inertia force and initial stress. The formulation is achieved by a systematic integration method which utilizes the orderly patterns of the polynomial shape functions. Results include the study of free vibration of circular arches under hydrostatic pressure; cylindrical and paraboloidal shells subjected to uniform and linearly varying stresses acting in the middle surface. Particular attention is given to the influence of initial stress on the frequency of free vibration and the extrapolated stress corresponding to zero frequency which yields the static buckling stress. It is found in some examples that the relation between the square of the frequency and the initial stress is not linear for the case that the fundamental vibration mode shape is the same as the critical buckling mode.**

## Nomenclature

| | |
|---|---|
| $a, b, t$ | = length, width, and thickness of the element, respectively |
| $D_m, D_f$ | = membrane and flexural rigidities, respectively |
| $f_i(\xi, \eta)$ | = shape function associated with $i$th D.O.F. |
| $H(\xi), h(\alpha)$ | = first-order Hermitian polynomial in $\xi$ and $\alpha$, respectively |
| $m, n$ | = mode number of vibration |
| $[k], [m], [n]$ | = element stiffness, mass, and incremental stiffness matrices, respectively |
| $\{p\}, \{q\}$ | = element load and displacement vectors, respectively |
| $R_1, R_2$ | = principal radii of curvature in $\xi$ and $\eta$ directions, respectively |
| $r_1, r_2$ | = inverse of $R_1$ and $R_2$, respectively |
| $S_1, S_2, S_{12}$ | = direct and shearing initial stresses in lb/in. ($S_{12} = S_3$) |
| $u, v, w$ | = displacements in $\xi$, $\eta$, and $z$ directions, respectively |
| $\alpha, \beta$ | = $\xi/a$ and $\eta/b$, respectively |
| $\gamma$ | = $EI/EA$ for the arch element |
| $\varepsilon_1, \varepsilon_2, \varepsilon_{12}$ | = direct and shearing strains in the middle surface of the shell |
| $\kappa_1, \kappa_2, \kappa_{12}$ | = changes of direct and twisting curvatures |
| $\xi, \eta, z$ | = curvilinear coordinate system |
| $\sigma_{\text{classical}}$ | = $0.605\,Et/R$ |
| $\omega$ | = frequency of vibration |
| $\lfloor\ \rfloor, \{\ \}, [\ ]$ | = row, column, and rectangular matrices, respectively |

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## Introduction

THE widespread use of shells as structural components of flight vehicles has stimulated many investigators to study various aspects of the structural behavior of shells. The predictions of natural frequencies of vibration and the critical values of middle surface compressive stress of shell structures have attracted considerable attention. It is, however, quite common that the shell structures vibrate while they are subjected to certain initial stresses. Undoubtedly, a combined study of